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ABSTRACT

These worksheets suggest activities through which secondary school students may be introduced to the ideas of simulation. All can be played as games to give students a feel for the situation modeled; most use coins or dice and a few use random digits. Directions are detailed for each of the 17 worksheets, with variations suggested for many. (MNS)

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SOME EXERCISES IN SIMULATION

(Pupil materials)

F. R. Watson

SE 036 394

Institute of Education
University of Keele
June 1980

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HOME EXERCISES IN SIMULATION: Pupil Material

These worksheets suggest activities through which pupils may be introduced to the ideas of simulation. They are intended for pupils of all ages in the Secondary School; though some will have more content for older pupils who may better appreciate the underlying theoretical ideas, all of them can be played as 'games' which will give pupils some 'feel' for the situation modelled. The order of the materials is, very roughly, that of increasing complexity of the situation simulated; most use coins or dice, a few use random digits. They are part of a larger introduction to this topic, intended for teachers, and older pupils (†) which also describes computer programs for performing these simulations.

Simulation is an attractive topic, providing game-like activities likely to interest pupils, illustrating the important ideas of probability and modelling, and showing how mathematical models can be used in practical problems clearly relevant to everyday life. It is an excellent illustration of the power of mathematics, and of the benefit of taking thought before plunging into action. (C.P. Ornell quotes an excellent example: Simulation of a proposal for operating a minibus service in Adelaide indicated that it would not be profitable; a businessman who thought he knew better abandoned the project after one day's operation!)

This topic provides a fertile field for work with computers, but, particularly in the early stages, there is a place for performing pencil-and-paper simulations like those given here, in order fully to understand the way in which the simulation works. Indeed with younger pupils, it seems desirable to use physical objects to represent items such as cars in traffic queues, producing random effects by using coins and dice. Only when pencil-and-paper recording is seen as a short cut, and the speed and convenience of random digit tables appreciated, should the concrete approach be dropped with these pupils. If the plunge into abstraction is too sudden, there is the danger that the whole investigation will become a meaningless ritual, devoid of connection with the real world.

(†) Watson, F.R. A simple approach to simulation, Institute of Education, University of Keele, 1980. (price £1.50 per copy).

The ultimate test of understanding is whether pupils can construct their own simulation models of simple situations of similar types.

The materials here are suggestions; they have not been extensively tested, and teachers may wish to adapt them to their own circumstances rather than use them as they stand. (Photocopying or other reproduction of these materials by teachers for use with their own pupils is expressly permitted.

A table of random digits is included, and simple practical methods of generating random digits are also described briefly (p.[illegible]).

The worksheets are printed so that related material appears back to back, half size. The "templates" which relate to some of the worksheets are fullsize.

I should like to thank Miss Annette Brindley and Mrs C Lea for their help in the production of these materials. Any comments, criticisms or suggestions for improvement will be welcomed by the author.

F.R.W. May 1980

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Note: Ways of Producing Random Digits

Perhaps the simplest way of producing a set of random digits from the set $\{1, 2, 3, 4, 5, 6\}$ is by rolling a die. Similarly a coin can be tossed to produce random sequences of '0's' or '1's' (eg 0='head', 1='tail').
As an alternative to a cubical die, random digits from the set $\{1, 2, 3, 5, 5, 6\}$ may be obtained by labelling the 6 faces of a hexagonal pencil or Ballpen. Tables of random numbers, like that on p 14 are also available, whilst for computer simulations the random number generator on the computer itself may be used. This note suggests ways in which you can generate your own random digit sequences for different purposes using

(i) a modified cubical die

or (ii),(iii) a calculator.

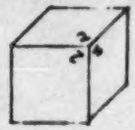
(i) Cubical die

(a) A cube has 8 vertices, which may be labelled 1 - 8. Using the rule 'select the uppermost vertex nearest the eye; in the case of a tie choose the R.H. vertex', random digits may be selected from the set $\{1, 2, \ldots\ldots 8\}$.
In a board game, arguments might arise about the correct choice of vertex (in which case an umpire might be needed).
In a simulation the choice is not important (provided that the experimenter does not deliberately choose an outcome to provide a 'freak' result).

(b) Similarly the 12 edges may be labelled, allowing choice of random digits from $\{1, \ldots\ldots\ldots 12\}$ with the rule - "take the uppermost edge nearest the eye".

(c) The 24 face corners may be labelled with the rule - "take the vertical face nearest the eye, and the number in top right hand corner of the face". (Selection from $\{1, \ldots\ldots 24\}$).

(Since there are just 24 rotational symmetries of the cube, this is the largest set of random digits obtainable in this way).

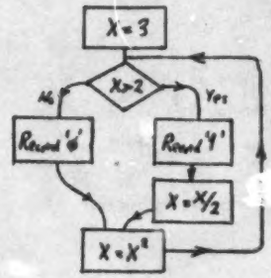


(a) Vertex-labelled cube, displaying '2'

(b) Edge-labelled cube, displaying '4'

(c) Face-corner labelled cube, displaying '19'

(ii) Random Sequence from $\{0,1\}$ using a calculator

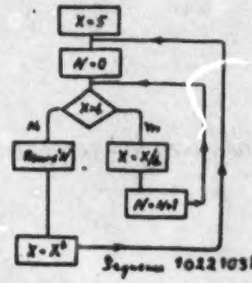


(i) enter 3 as X
(ii) if $X>2$, record '1' and divide by 2 else record '0'
(iii) square X
(iv) goto step (ii)

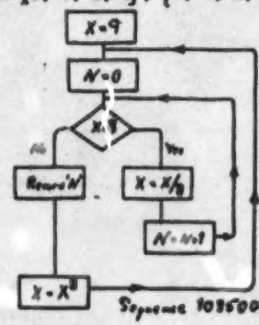
The sequence is 'pseudo-random' and in effect gives the binary value of x where $2^x = 3$
(Consider $3 = 2^x = 2^{1+y}$
Then $2^y = \frac{3}{2}$ so that
$2^{2y} = (\frac{3}{2})^2 = 2.25 = 2^{1+z}$
where $z = \frac{2.25}{2}$)

Sequence is 1100101011....

Since x is known to be irrational, the sequence will recur only because of the finite accuracy of the calculator used. (eg. it may work to only 8 decimal digits).
In practice, it is found that this pseudo-random sequence contains '0's' and '1's' in approximately equal proportions, and in other ways behaves as a genuinely random sequence.
Modifications will produce random sequences from $\{0, 1, 2, 3\}$, $\{0, 1, 2, 3, 4, 5, 6, 7\}$



Sequence 10221031....



Sequence 10300021....

Work Sheet1 GreenX (Crossing the Road)

Problem

A pedestrian wishes to cross a busy road from X to Y.
Traffic is coming from both directions L and R and so he can cross safely only if there is no vehicle in either of the road sections AY, XB in the diagram (XB = AY = 100ft., say).
Traffic arrives randomly from both directions; the probability that there will be a vehicle in AY during a given period of 10 secs (the time it takes for the pedestrian to cross) is p_L.
Similarly the probability that there will be an arrival from the right in XB, is p_R. What is the chance, in any time period of 10 sec, that he will be able to cross?

Simulation

Carry out the simulation as follows:-

(a) Suppose $p_L = p_R = 0.3$ (light traffic)

(i) Choose a random digit. If it is 1,2, or 3, place a counter in the section AY to represent a vehicle. If it is 0, or 4-9, leave AY empty.

(ii) Choose another random digit. Again if it is 1,2,or 3 place a counter in the section XB, otherwise leave XB empty.

(iii) If both AY and XB are empty record a successful crossing. Otherwise record 'unsuccessful'.
Carry this out 20 times (representing 20 x 10 sec time intervals), to find the number of successful and unsuccessful crossings.
Estimate the chance of a successful crossing.

(b) Repeat the whole simulation with $p_L = p_R = 0.7$ (rush hour) e.g. let '1','2','3','4','5','6','7' represent 'vehicle arrives', '8','9','0' = 'no vehicle'. You will need two separate random digits to decide what happens in the two road sections.

(c) Find out how to calculate the probabilities of a successful crossing in cases (a) and (b) above, and compare with your results.

Worksheet 2

Petrol Pumps

Description

Cars are arriving at a petrol station which has only one pump in operation.
If there is a car already at the pump, they do not queue but go elsewhere and are lost as customers.
We suppose each car takes 60sec. to serve, and consider time units of 10 seconds, so that the service-time for a car is 6 'units'.
Suppose cars arrive at an average of one a minute (so that the probability p of an arrival in one time 'unit' is given by $p = \frac{1}{6}$).
(In this case the average gap between vehicles is 6 units, equal to the service time).

Problems

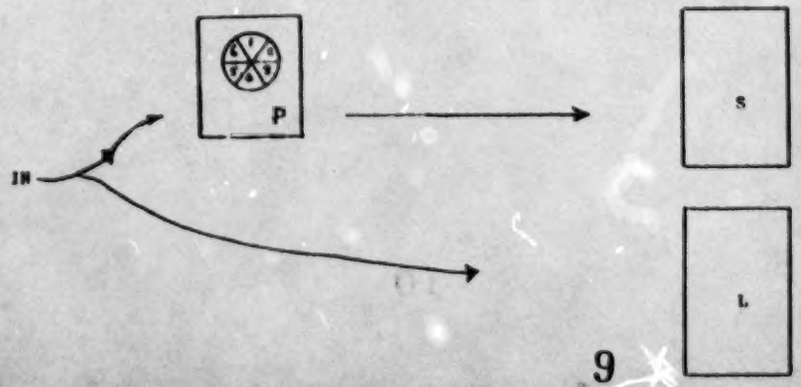
a) What proportion of customers will be served, and what proportion will go elsewhere?

b) How would the answer to (a) change if the traffic density is doubled? (i.e. 2 vehicles arrive per minute; $p = \frac{1}{3}$, where p is the probability of an arrival in a single time unit).

c) Suppose the garage installs <u>two</u> petrol pumps. What difference does this make to the situations described in (a) and (b)?

Method

Use two sheets of paper labelled S and L (to keep a record of the customers served and customers lost respectively), and another sheet P, marked with a 'clock' (a circle with 6 subdivisions, as shown) to represent the petrol pump. A counter placed on P will represent a car being served. In each time unit (10 sec) the clock finger moves on one subdivision. (Use a pencil or another counter as the finger). When the clock shows '6', the car is ready to leave.



2(contd.).

(i) At the beginning of each time interval, throw a die.

(ii) If '6' is obtained (representing 'car arrives') a counter is taken.

(iii) A counter representing a new vehicle may be placed on sheet P if P is empty ("pump free"), otherwise it is placed on L ("customer lost").

(iv) If there is a counter on P, the clock 'finger' is moved round one place. When the finger shows '6', the car is ready to move to S ("customer served").

(v) End of time unit.

Continue from step (i) until 100 throws of the die have been made. At the end, record the number of counters on S and on L (ignore counters still on P).

For (b):- alter step (ii) to give "vehicle arrives if dice shows '5' or '6'."

For (c):- use two cards P_A and P_B to represent two pumps

In (iv) the clock 'fingers' for any counters on P_A or P_B are moved one place clockwise in each time unit.

Modifications

(d) Use more cards to represent installation of extra pumps.

(e) 'Speed up the service time' by installing a high-speed pump (i.e reduce the petrol pump 'clock' to 4 subdivisions)

(f) Alter the density of traffic - e.g. by using

'4', '5' or '6' = vehicle arrives ($p = \frac{1}{2}$ - i.e. 3 vehicles/minute)

or '3', '4', '5' or '6' = vehicle arrives (4 vehicles/minute).

Note

Tables of random digits may be used instead of dice.

E.g. '9' or '0' = "vehicle arrives" gives a traffic density of 1 vehicle per 50 seconds, ($p = \frac{1}{5}$).

Worksheet 3 Pinball

Fig(i) is a pinball machine - you sometimes see them in Amusement Arcades.

How to Play

You need a counter (or some other 'marker') to show where the (imaginary) "ball" is, and a die to decide where the ball goes next. Keep a record of your score in each game.

1. Place the counter on square A (start).
2. Roll a die. If the result is even, go to square C (end of game).
   If the result is odd go to square B and add 10 to your score.
3. From B - roll a die again. If the result is even, go to square C (end of game).
   If the result is odd, go to square D and add 10 to your score.
4. From D - roll a die. If you get a '6', go to square A and start again.
   If you get anything else, go to square C (end of game).

Record your score for the game. What do you think your average score in a lot of games would be?

Play 10 games in all and record your score for each. Work out the average.

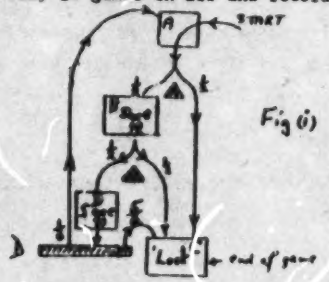


Fig (i)

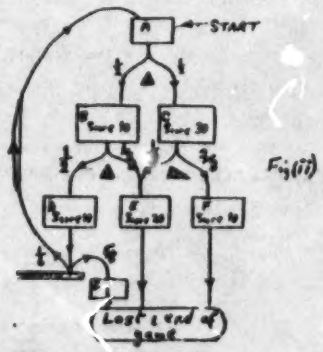


Fig (ii)

What to do Next

(A) Use the diagram of the more complicated pinball machine shown here. Write down the rules you will use to decide where to go next from each box, and how to alter your score at each stage. (To help you - you might decide at C:- "throw a die. Score of '1' or '2' → go left to E; score of '3,4,5' or '6', go right to F").

Play one or two games and record your score for each.

Estimate your average score in a lot of games.

Now try playing (say) 10 games and work out your average score for all 10.

(B) Make up your own pinball machine diagram and play some games on it. Don't make it too complicated; remember you have to predict what the average score in a game will be!

Worksheet 4

Rumour

A rumour spreads among a group of 6 friends like this:-
When one person hears the rumour he telephones one of the other 5, chosen at random, to pass it on. When he has done this, he takes no further interest in the rumour. The second person passes on the rumour in the same way to just one other person chosen randomly; if the message is passed to someone who has already heard it, both individuals stop talking about it, and no further people are informed.

Problems

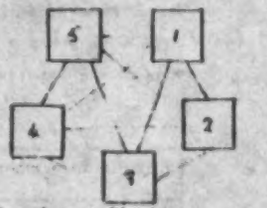
Is it likely that all 6 will get to hear of the rumour?
On average, how many of the 6 will get to hear of it?

Simulation

We can choose one person at random in a group of six by rolling a die.

Method

Use a copy of the diagram and a supply of counters.
Roll a die; whatever number it shows, place a counter on that square (to show a new person has been told the rumour), unless the number is the same as the previous one (he wouldn't tell the rumour to himself!).
If you get two counters on a square it means someone has heard the rumour twice - so it dies out. (A person is allowed to tell the rumour straight back to the person who told him - perhaps he is very forgetful!).
At the end, record how many squares have counters on (i.e. how many people have heard the rumour).
Repeat the experiment several times - put the results in a table like this



Expt. number	1	2	3	4	5	6	.....
No. of people hearing rumour	4	6	2	3	...	...	...

Worksheet 6

Radioactive Waste

Description

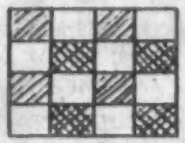
Suppose a new radioactive substance ('Dalekium') has been discovered with a "half-life" of 10 minutes. (This means that, if at first there is 100g of the substance, after 10 minutes only 50g will remain, and after a further 10 min., only 25g will be left - and so on - because it 'decays' to form other substances).

Problems

If we start with 100g, how much will be left after one hour?
How long will it take for the amount present to fall below the "danger level" (of 0.001g remaining)?

Simulation

We simulate the process of decay by scattering counters at random on a board, so they fall on a shaded square with probability ½ (each counter which falls on a shaded square corresponds to an atom which disintegrates during a 10 min period). Those counters which have fallen on a shaded square ("atoms which have disintegrated") are removed, and another '10 min cycle' of the simulation is performed, by scattering the counters again.



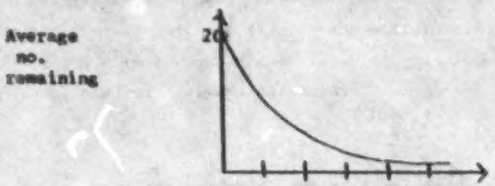
Method

Place a piece of A4 paper, with alternate squares shaded (like a chessboard - see fig.) inside a box lid. (A box in which A4 paper is supplied is ideal).
Place 20 counters (or 1p coins) in the box lid and shake them up (or side ways).
Eliminate any counters/coins whose centres lie on a shaded square.
Repeat until all counters are eliminated, and record your results at each stage:-

Time interval (measured in 10 min ('half-life') periods	0	1	2	3	4	5	6
No. of counters ("atoms") remaining	20	11	5	.....			

6 (cont).

Repeat the experiment two or three times more, and work out the average number remaining after 1 throw, 2 throws, 3 throws,.....
Plot a graph of average number remaining against number of throws.



It should look something like the smooth curve in the diagram - but since the number of 'atoms' in your simulation is small, and the amount remaining must be a whole number of atoms, the graph of results is likely to have 'kinks' in it.

Note.

100 g of any substance contains a very large number of atoms - about 1000000000000000000000 (23 zeros) in fact, so that the 'kinks' in the graph representing decay of an individual atom are too small to be noticed.
(Alternatively coins may be tossed and those which fall 'heads' removed at each stage. The method using counters is likely to be quicker and more convenient - and not so many coins are needed!)

Modification

Simulate the behaviour of another isotope which decays more slowly:
(p (decay in 10 min period) = ½) i.e. ½ of the original amount is left after 10 minutes.

Method

Use the counters as above, eliminating only those which fall with their centres on a doubly shaded square (p = ¼)

Again record results in a table, as above.
(If coins are used, before eliminating those which fell H, toss them again and eliminate only those which fall H a second time - $P_{HH} = ¼$).

4 (cont)

Variations

Suppose now there are 12 people in the group.
Use 2 dice. If the first one shows 'even', use the number on the second die to decide who hears the rumour next. If the first one shows 'odd', add 6 to the score on the second die in order to decide. (Ignore cases where a person tells the rumour to himself - i.e. you get the same result twice running).

1	2	3
4	5	6
7	8	9
10	11	12

Put a counter on a square to show that the person has heard the rumour.
Stop when you get two counters on the same square.
Repeat the experiment several times and make a table as before.

Experiment number	1	2	3	4	5	6	7 .....
No. of people hearing rumour							

Worksheet 5

Christmas Pudding

Description and Problem

Suppose a Christmas pudding is shared between six people so that all have equal portions. Hidden in the pudding there are six coins. Is it likely that every person will get a coin? How many people, on average, will get no coins? How many will get more than one?

Simulation

Use a dice to decide, for each of the 6 coins, which portion of the pudding it goes into.

Method

(a) Use a copy of the circle shown divided into six sectors representing the six portions. Roll a die, and place a counter on the sector corresponding to the number shown. Continue until 6 counters (6 coins) have been placed in sectors.

How many sectors are empty ("have no coins")?

How many sectors have more than one counter ("more than one coin")?

Repeat the experiment several times and record your results in a table like this:-

(Each column represents a new experiment)

Expt. no.	1	2	3	4	5 ...
No. of empty sectors					
No. of sectors with more than one counter					

Variations

(b) Use random digits to simulate sharing a 10-coin Christmas pudding between 10 people.

Choose 10 digits in turn and place a counter on the corresponding square each time.

1	2	3	4	5
6	7	8	9	10

(c) Use 2 dice - or instead use one die and a penny - to simulate division of a 12-coin Christmas pudding between 12 people.

Method

Roll the first die. Then roll the other die (or toss the penny). If the second die shows even, (or the penny shows 'H') add 6 to the score of the first die; otherwise add 0 to the score of the first die.

Notice that the proportion of people who get no coin at all is about the same in all three cases (a), (b) and (c).

Worksheet 9

Birthdays

What is the chance that in a class of 30 children, there are two people with the same birthday? The answer is rather surprising.

(1) You might like to try it for your class - list everyone's birthday. Are there two the same?

Now use either (2) or (3) below:-

2. Using Dice

To make the problem simpler we will suppose that there are 12 equal months, each of 30 days. (This would give 360 days in a year - not far out).

(A) Roll 2 dice, to decide the birth month. Roll first one and then the other. Use the first die to decide whether the result is Jan-June ('1','2','3', on 1st die) or July - Dec ('4','5','6' on first die) and the second to decide which month, (from 1-6)

Example

If you throw '4' first time (July-Dec) and '2' second time, (Feb. or August), the result is August.

(B) Now find the date within the month. Roll 2 dice in succession.

(i) If the first die score is '6', roll it again.

(ii) Take first die score and subtract one.

(iii) Multiply the result by 6 and add score on second die.

Example

Scores are '3','5'. Date is (2 x 6) + 5 = 12 + 5 = 17(th).

Note. A score of '1','1' would give (0 x 6) + 1 - i.e. 1st of month and of '5','6' would give (4 x 6) + 6 i.e. 30th of month.

(Remember: score of '6' on first die is ignored - throw again).

(C) Write down the 'birthday' obtained in this way - in our case Aug 17th.

(D) Repeat this until you have 30 randomly chosen 'birthdays' (but stop if you get two the same before you have done 30).

Do the whole experiment several times.

What do you think is the correct value for N in the following statement?

"If there are more than N people in a room, the chance that two of them have the same day as their birthday is $> \frac{1}{2}$"

9.(cont).

3. Using Random Digits

Suppose each of the days of the year is labelled, from 1 to 365 (we ignore leap years).

(A) Choose 3 random digits to give a 3 figure random integer R.
If $1 \leq R \leq 365$, R gives the date.
If $501 \leq R \leq 865$, take the date as (R-500).
(If R lies outside these ranges, try again)

(B) Repeat this process to assign birth dates to 30 people (or until 2 birth-dates coincide).
Do the whole experiment several times.
What do you think is the correct value of N in the following statement?
"If there are more than N people in a room, the chance that two of them have the same day as their birthday, is $> \frac{1}{2}$".

Worksheet 7

Collector's Problem

A cereal manufacturer provides a small plastic badge with each packet of cereal. If there are 6 different types of badge and they are randomly distributed one to each packet, how many packets of cereal must I buy, on average, to obtain a complete set of the badges?

Simulation

(a) Roll a cubical die and take the number displayed as the number of the next badge collected. Continue rolling the die until all of the numbers '1,2,.....6' have appeared. Record the total number of throws required.
(It may help to write a list of numbers 1 - 6 and cross them off as they appear.)

(b) Repeat several times for comparison. Find the average number of throws required.

Alternatives

(a) Simulate collecting a set of objects by tossing 3 coins, until each of the possible 8 results has appeared.

HHH HHT HTH HTT
THH THT TTH TTT

(b) Simulate collecting a set of 8 digits by using a cube with corners labelled 1,2,.....8.
(Selection rule: "Choose the uppermost corner which is nearest to your eye. In the event of a tie, choose the right hand one").

(c) Simulate collecting a set of 12 objects using a cube with edges labelled 1,2,.....12.
(Selection rule: "Choose the uppermost edge which is nearest you; in the event of a 'tie', choose the right hand one").

(d) Simulate collecting a set of 10 objects (or n objects where $n < 10$) using random digits. (For $n < 10$ ignore all digits outside the appropriate range).

Worksheet 10

Road Up

Description

Temporary traffic lights are installed to control one-way traffic when a section of road running North/South is under repair (see fig.). Because vehicles cannot clear the one-way section instantaneously, there will be a lag between the halting of traffic in one direction, and the release, by a 'green' signal, of traffic in the other. The rate of flow of traffic in the two directions may differ - (e.g. at rush hour).

Problem

How should the timing sequence for operation of the traffic lights be adjusted for best results?

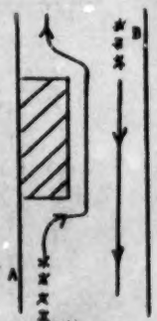
Simulation

Suppose the time lag between 'green' signals (to allow traffic to clear the one-way section) is 5 seconds, and that just two vehicles can pass through the section in 5 seconds. Suppose also that N-bound traffic arrives at the average rate of 6 vehicles per minute (1 every 5 sec.) and traffic travelling S, at 1 per minute (1 every 5 sec.).

(a) Consider first the traffic light pattern:

I	N bound lights green,	5 sec.
II	Both lights red,	5 sec.
III	S bound lights green,	5 sec.
IV	Both lights red,	5 sec.

and use 5 sec. as a unit time interval.



Method

In each time period:

(i) Toss one coin (for N bound traffic). If coin shows 'H' place a counter at A (to show "vehicle arrives").

(ii) Toss 2 coins (for S bound traffic).
If result is HH, 2 vehicles arrive at B.
If result is HT or TH, 1 vehicle arrives } at B.
If result is TT, no vehicle arrives

(In brief, number of vehicles arriving = number of 'heads' obtained).

Worksheet 11

Road-Junction

Description

The diagram below represents a one-way traffic lane joined by a feeder road B, (such as might be found on a dual carriageway). In each 10 second period a car approaches from the direction A with probability P_A and, independently, a car approaches from B with probability P_B. Cars at B can exit only if there are two successive gaps in the A traffic; otherwise they form a queue.

Problem

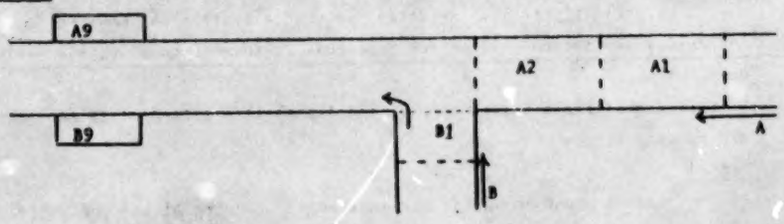
How does the queue length at B depend on traffic density?

Would installing traffic lights help?

Simulation

Take $P_A = \frac{1}{2}$, and $P_B = \frac{1}{6}$. We use a coin to decide arrival of traffic from A, and a die for traffic from B.

Method



Use a copy of the diagram, which represents a road junction with A and B traffic coming from the directions as shown. B traffic turns left into the traffic stream (if no A traffic is present - A has right of way). The regions A9, B9 are used to hold counters representing A and B vehicles, respectively, which have passed through the junction.

(i) If there is a counter in A2, move it to square A9 (A traffic which has gone past the junction).

(ii) If there is a counter in A1 move it to square A2 (traffic moving along main road).

(iii) Toss a coin. If the result is 'H', place a counter on square A1 to represent "car coming".

(iv) Roll a die. If the result is '6', place a counter on square B1 (vehicle arrives from B) - to represent a queue forming.

(v) If A2 and A1 are both empty, allow one car from B1 to leave the queue. (Remove one counter and place on square B9 ('B traffic which has gone past the junction).
This corresponds to a 10 sec. period.

(vi) Go back to step (i),until you have done 30 steps of the simulation (i.e. '5 minutes').

At the end of the experiment, record results:

Number of vehicles from A (= no. of counters in A9)
Number of vehicles in B queue (= no. of counters in B1)
Number of vehicles from B (= no. of counters in B1 + number in B9).

Alternatively, write down the number of vehicles in B traffic queue (i.e. counters on B1) at each stage, and find the average queue length. (This gives a measure of the average waiting time for vehicles trying to escape into the main stream of traffic).

Modification

Suppose traffic lights are installed, so that 'A' traffic can move for 6 periods of 10 sec. and is then halted for 2 periods of 10 sec. to allow 'B' vehicles to move. *

Suppose 2 vehicles can pass through the lights in a period of 10 sec. when they are showing 'green'.

Method

(i) Toss a penny to decide arrivals at A. (Ignore A1 and put vehicle counters on A2).

(ii) Roll a die to decide arrivals at B.

(iii) Add vehicles to queues at A and B.

(iv) Reduce queues at A and B according to traffic light setting. (N.B. queue can reduce by 2 if there are sufficient vehicles in the queue).

(v) Go back to stage (i) until 32 steps of the simulation have been carried out.

Compare number of arrivals at A,B (counters on A9 + queue A2 and counters on B9 + queue B1, respectively) - they should be similar to earlier findings.

Compare queue lengths and average queue lengths under the new system - is it an improvement?

*You may wish to experiment with other traffic light settings to see if you can improve further on this.

Place counters at B to represent vehicles.

(iii) If traffic lights are 'green', waiting vehicles may move forward. At most two vehicles may pass through the narrow section in one time period. Any which cannot pass through remain in the queue. (Those which have successfully passed through should be kept - or at least counted).

(iv) Traffic lights at A and B change according to the pattern given.

Continue the simulation for 24 time periods, by returning to step (i). At the end of this activity, examine the size of the queues which have formed, and the numbers of vehicles which have successfully passed through the narrow section in N and S directions.

(b) Now change the traffic light pattern to:-

I	N green,	10 sec.
II	Both red,	5 sec.
III	S green,	10 sec.
IV	Both red,	5 sec.

and repeat the simulation. Is this pattern an improvement?

(c) Repeat for the pattern

I	N green,	25 sec.
II	Both red,	5 sec.
III	S green,	25 sec.
IV	Both red,	5 sec.

(d) Repeat for the pattern

I	N green,	15 sec.
II	Both red,	5 sec.
III	S green,	30 sec.
IV	Both red,	5 sec.

Which seems the most appropriate pattern?
Can you suggest a better one still? Try it out!

Worksheet 8

Christmas Tree Lights

Description and Problem

I have a set of Christmas tree lights containing 10 bulbs. The chance that any single bulb will fail during one year's use is $\frac{1}{36}$. I have two spare bulbs, so that when three bulbs have failed, the set will be useless - as spare bulbs are now unobtainable.

How many years can I expect the set to last?

Method

Place 10 counters on the table to represent the 10 bulbs, and have two others ready to represent the spares. For each counter in turn, throw 2 dice to decide if that bulb fails during the year. (If you get 'double-6' assume 'bulb fails' and remove that counter - it represents a 'dud' bulb). Continue until all counters have been tested; that represents one year's use.

(The spares are not tested, as they are not in use).

Now, if you have spare bulbs left, make up the set to 10. If you cannot, the set is now useless - record how long it lasted. If you can make a complete set of 10, continue the experiment for another "year".

Try the whole simulation several times over, to compare results.

Modifications

(a) Use cheaper, poor quality bulbs - e.g. 'double-5' or 'double-6' = 'failure', (so that P(failure) = $\frac{1}{18}$).

(b) If the bulbs are cheaper we may be able to allow more spares - so allow 4 spares (i.e. stop when 5 bulbs have failed).

Example	Year number	1	2	3	4	5	
	No. of bulbs failing during year	1	0	1	2	1	(set fails after 5 years)

(c) Use a 'super quality' bulb with a low probability of failure (say P = $\frac{1}{72}$) but only one spare bulb. (So that the set is useless after 2 failures).

(For P = $\frac{1}{72}$ use 2 dice and a coin. Bulb fails only if coins shows 'head' and dice show 'double-6'.

Alternately random digit tables may be used, e.g.

random digits 01 (or) 02 ⟶ 'bulb fails'; this gives P(fail) = $\frac{1}{50}$

Worksheet 16

Foxes and rabbits

An island is thickly covered with grass and supports a large population of rabbits; let the number of rabbits at a given moment be X. If there were no checks on population growth, X would increase exponentially, (ie according to the "law of compound interest"), since the number of rabbits born in a period of time would be proportional to the existing population. Suppose, though, that a population of Y foxes also lives on the island, feeding on the rabbits. If we assume no rabbits die of old age, the rate of decrease of rabbits will be proportional to the product XY. Similarly, we assume that the rateof growth of the fox population is proportional to XY (more foxes, more cubs; more rabbits, more food for them). The foxes may die of old age, and then the rate at which they are reduced will be proportional to Y. We shall measure time in months, and choose the units of X and Y so that the equations determining X and Y become

$$\Delta X = \tfrac{1}{2}X - \lambda XY \quad \ldots\ldots\ldots (1)$$

(change in X per month = {increase due to births} + {Decrease due to foxes})

$$\Delta Y = -\tfrac{1}{2}Y + \lambda XY \quad \ldots\ldots\ldots (2)$$

(change in Y per month = {decrease due to deaths} + {increase due to births})

where $\lambda = \frac{1}{70}$.

It can be deduced from these equations that both populations will oscillate in number about a mean of 35, as shown in the diagram:-

(When there are a lot of rabbits, the foxes will multiply and so begin to reduce the rabbit numbers; as this happens, there will be less food for the foxes, so their numbers will diminish in turn, allowing the rabbits to increase again).

Simulation (adapted from Thompson, 1979*)

We use a 'stochastic' method (ie one based on chance) which mirrors the equations 1 and 2 in terms of probability. Because of the chance element, there is a possibility that one of the populations will die out.

Method

Use a piece of A4 paper with 32 rectangles, shaded alternately inside a box lid. (cf Worksheet 6, p 43)

(* Thompson, J MT,An evolution game for a prey-predator ecology, Bull I.M.A. 15, 7 July 1979)

Take F black counters ('foxes') and R white ones ('rabbits') - suitable starting values are F=10, R=10.

1) Shake all the counters in the box lid.
   A counter is 'on' a square if its centre is inside the square.
   If any square contains more than 3 counters of any one colour, 'excess' counters are removed. ("deaths by overcrowding").
2) Any black counter which is 'on' a black square is removed (Fox dies, $p = \frac{1}{3}$)
3) Any white counter on a black square is replaced by two white counters (rabbit reproduces, $p = \frac{1}{3}$)
4) For each pair of black and white counters which are together on a white square, two black counters are added and two white counters are removed (two foxes are born, and two rabbits eaten)

   (Note. Step 4 is not repeated for black counters representing newly-born foxes)

The numbers of foxes and rabbits at the end of step (4) are recorded and the whole cycle repeated.

Note. In step 4 it is found experimentally that when X=Y=35 (the 'equilibrium" values) a board of 32 squares gives about 17 interactions between foxes and rabbits, agreeing with the value $\lambda = \frac{1}{49}$ in equations (1) and (2).

Worksheet 17

Diffusion of an insect pest (*)

The diagram (fig (i)) shows an area of land near a port P where an insect pest has been accidentally introduced. The area shaded diagonally represents barren ground, where the pest cannot live; the area marked with dots is the sea. The pest spreads by flying and is most likely to spread NE because of the prevailing wind.

Simulation

Infection spreads from an infected square to adjacent squares according to the scheme shown in fig (ii). Two dice are thrown; the sum of the scores indicates the next square to be infected (unless it is sea or barren land).

Method

1) Mark the port with the number 0 (generation 'zero') Throw the two dice and determine the next infected square; mark this 1 (generation '1')
2) For each square infected by generation 'n' (ie 0, 1, 2, 3, 4.... n) determine the 'next square infected', using fig (ii) - mark all such squares generation '(n+1)'.
3) Repeat step (2) until n=20, say.

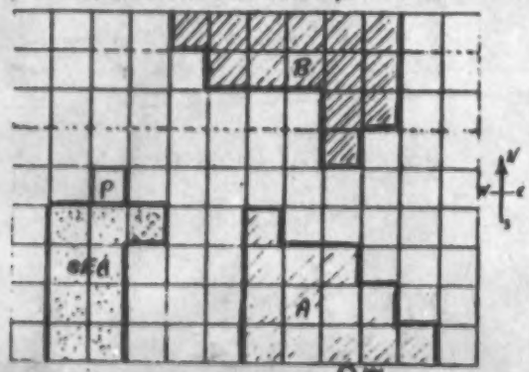
Note: many choices of 'next infected square' will be wasted because the square is already infected, or is barren or sea.

Modification

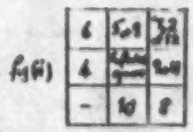
Tables of random numbers may be used instead, with fig (iii), to determine 'next square infected'. eg, if the 2-digit random number chosen is 43, then infection takes place [illegible]

Notice the effect of the 'corridor' between the mountain regions A and B, and also how slowly the infection spreads Westwards.

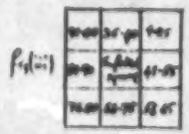
(*based upon an idea of K Selkirk).



fig(i)



fig(ii)



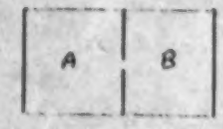
fig(iii)

Worksheet 14

Ehrenfest Diago Game - a model of gas diffusion

Problem

Two equal compartments are separated by a partition in which is a small hole. (See diagram (i)). At first, A contains a large number of molecules of a gas and B is completely empty. Molecules in A tend to move into B; any which have moved into B tend to move into A again. After a long time period, we expect the molecules to be fairly evenly divided between A and B ("equilibrium").



(i)

Board B

(ii)

Simulation

We choose a molecule at random and transfer it to the other compartment. At first the movement will be almost entirely in the direction A ⟶ B, but gradually the rate of transfer in the other direction will increase as the number of molecules in B increases. We suppose there are 36 molecules in A to begin with.

Method

Each of the 36 squares on the board B is labelled to show the result of throwing 2 dice; e.g. (3,6) represents '3' on first die and '6' on second. (P_{ij} (ii)).

At first B is empty (corresponding to the empty compartment B).

Each time the two dice are thrown a counter is placed on the corresponding square of B, unless there is already one there - in which case that counter is removed from B. The number of empty squares on B is recorded. At first this will decrease sharply, but as the 'equilibrium position' is approached, the rate of decrease will slow, until in the later stages random oscillation about equilibrium will occur. It is instructive to draw a graph of number of squares against number of throws (i.e. 'diffusion' against 'time').

Worksheet 15

Drunk!

Description

A drunken man leaves a public house P at '5' in the diagram below. His home H is at '9', and if he arrives there, his wife will put him to bed.

At '1' is a Police Station, S; if he arrives there he will be put in a cell overnight to sober up. He takes half a minute to cover each unit of distance, at the end of which he falls to the ground and takes another half a minute to get up. Each time he begins to walk, he is equally likely to go in either possible direction.

Problem

Is he more likely to end up in jail or at home?

How long, on average, does it take for him to get there?

Method

(a) The time between successive tosses of a coin represents one minute. If the coin falls 'H' he moves one step towards home (Right). If it falls 'T' he moves one step Left (towards jail): ($P_L = \frac{1}{2}$, where P_L = probability of moving Left: if you use a die, take 'odd number' = 'H' = move R, 'even number' = 'T' = move L).

(i) Start on square '5'. Place a counter there to represent the man.

(ii) Toss the coin and move the counter according to the rules.

(iii) If the counter has not reached square '1' or square '9', return to step (ii). If it has, record the number of moves taken and where he ends up (in a table like that shown below) and return to step (i) until you have performed the simulation several times.

Specimen Table of Results

Simulation Number	Finishing Point (S or H)	No. of Moves
1	S	10
2	S	4
3	H	10
4	S	18

(contd).

Modifications

(b) Suppose the street slopes down hill from S to H, so he is more likely to move R at each step.

(With a die, take '5' or '6' = move L, so $P_L = \frac{1}{3}$

With a coin, toss twice and use 'HH' = move L; 'TT', 'HT' or 'TH' = move R, so that $P_L = \frac{1}{4}$).

As before, find whether he is more likely to end up at S or at H. When he starts from P, what is his chance of reaching H?

Note. Tables of random digits could be used instead:

e.g. in (a) Odd digit ⟶ "move L"; even digit ⟶ "move R"

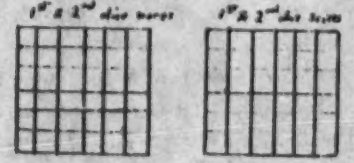
in (b) Digit = '0,1,2' ⟶ "move L"; any other digit ⟶ "move R"

(so $P_L = \frac{3}{10} = 0.3$).

(contd.)

Extensions

(a) Use 3 dice and 216 ($=6^3$) squares arranged in 6 sets of 36.



(216 counters in A, and none in B, at the start).

, etc. .....

3rd dice score 1 — 3rd dice score 2

(3rd die tells you which square to choose. 1st + 2nd tell you position in that square)

or

(b) Use pairs of random digits and a 10 x 10 square for B, or triples of random digits and a board numbered 1 - 200, (or even 1 - 1000).
(The latter is best as a class activity)

Worksheet 12

Inflation

For this simulation you need a calculator.

Problem (1)

The value of a certain house increases by 15% every year. How long will it take to treble in value?

Method

If the rate of increase is 15% each year,

after 1 year, £1 becomes £1 x 1.15 = £1.15

after 2 years, £1 becomes £1 x $(1.15)^2$ = £1.3225, and so on.

Use a calculator to find how many times you must multiply by 1.5 to convert £1 into £3.

Problem (2)

What rate of inflation will double the value of a house in 3 years?

(The answer is not 33⅓%).

Method

By trial $1.15^3 < 2$, so 15% is not enough.

Also $1.3^3 = 2.197$, so 30% is too much.

Use trial and error to find the % inflation needed.

Problem (3)

An isolated island has plenty of grass and a population of rabbits which increases at a constant rate. (The number of rabbit-births and deaths is proportional to the number of rabbits present - assuming that famine, disease and destruction by other animals do not intervene).

If the population increases 10 fold in 10 years, how much will it increase in one year and in 15 years?

Method

The same as for problem (2).

Problem (4)

Radioactive dalekium isotope has a half-life of 10 min, i.e. of a sample of a 100g of Da, only half will remain 10 min. later, the rest having disintegrated, while after 20 min., only 25g will remain.

Find the amount remaining after 1 hour and also after 1 minute.

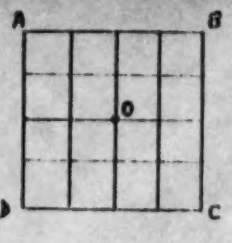
How long will it take for the amount to fall below the 'danger level' (0.001g remaining)?

Method

See (2) and (3) above.

Worksheet 13

Atomic-Pile (simplified version in 2 dimensions)



Problem

The diagram represents an atomic pile O, surrounded by a layer of shielding A B C D. Neutrons are emitted from the pile; the shielding is intended to absorb them before they leave the square A B C D.

How effective is the shielding?

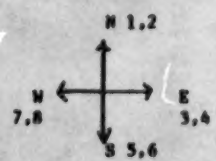
Is it thick enough?

Simulation

We assume that a neutron starts from O and each second moves exactly 1 unit either N, S, E or W, (or stays where it is), and that if after 5 seconds it has not escaped, by reaching the edge of the square A B C D, its energy is so low that it will be absorbed where it is.

(a) Start with a neutron at O. Choose a random digit and decide the movement of the neutron by these rules:

If digit is { '1' or '2', move N; '3' or '4', move E; '5' or '6', move S; '7' or '8', move W; '9' or '0', stay put }

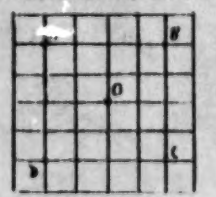


(9,0, stay put)

Trace the progress of the neutron for 5 moves (5 digits) to decide whether or not it escapes. Repeat for 10 neutrons altogether.

What fraction of the neutrons escapes?

(If you can, pool your results with those of other pupils; you will get a better overall estimate since your group of neutrons may have been 'lucky' or 'unlucky').

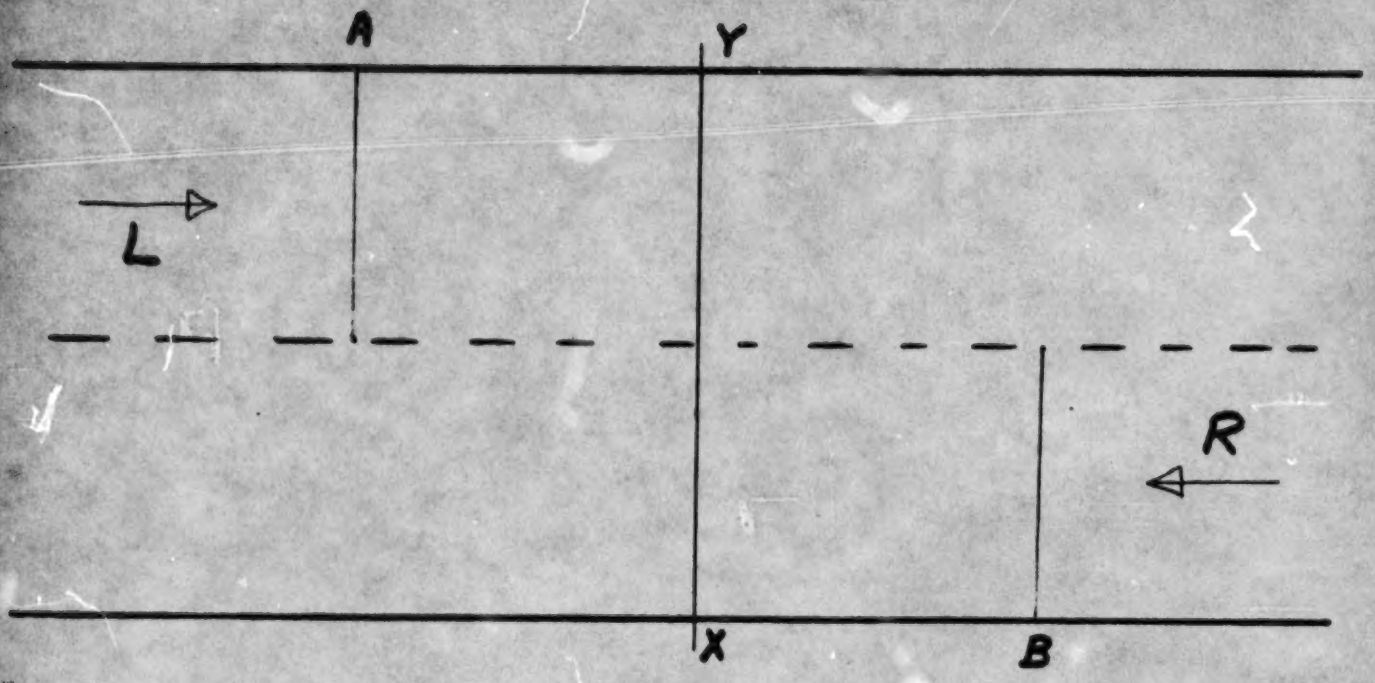


(b) Repeat the simulation using a thicker layer of shielding (a 6 x 6 square) tracing 10 neutrons as before, (see figure).

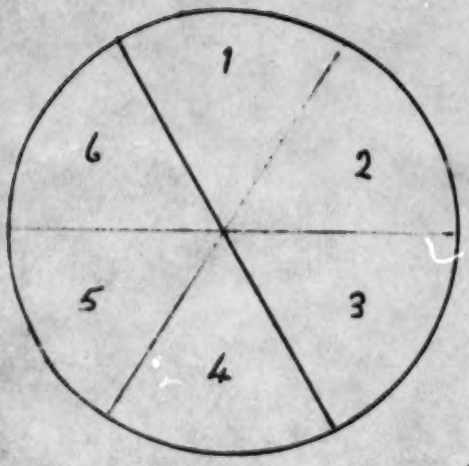
(c) Repeat (b) but now suppose the neutrons to be more energetic with a 'life' of 10 sec., (so trace their progress using 10 random digits instead of 5).

Is the thicker shielding effective?

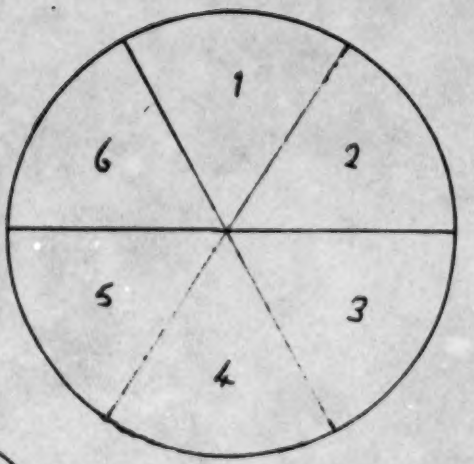
Worksheet 1(a)



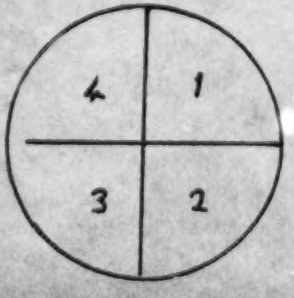
Worksheet 2(a)



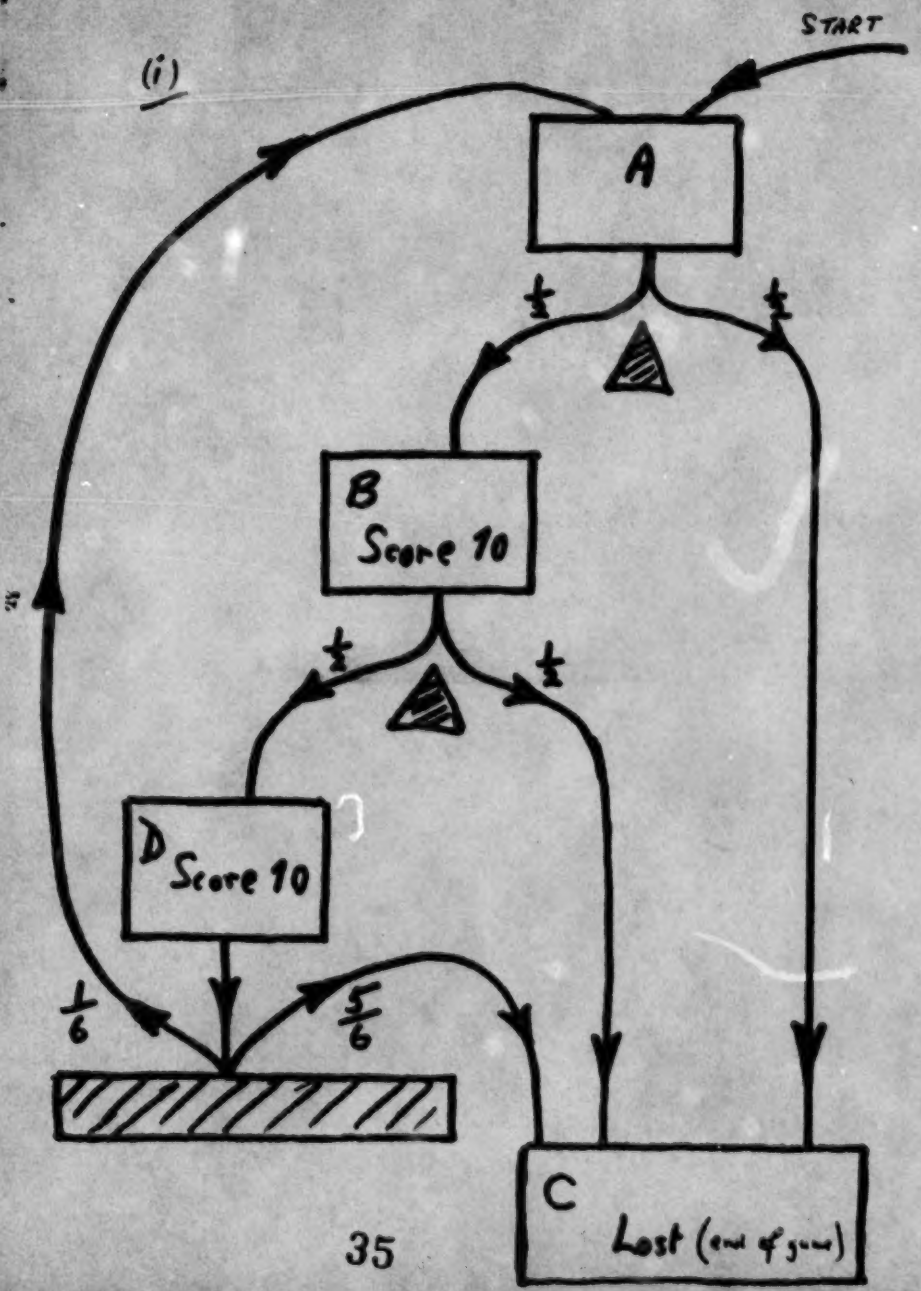
P_4

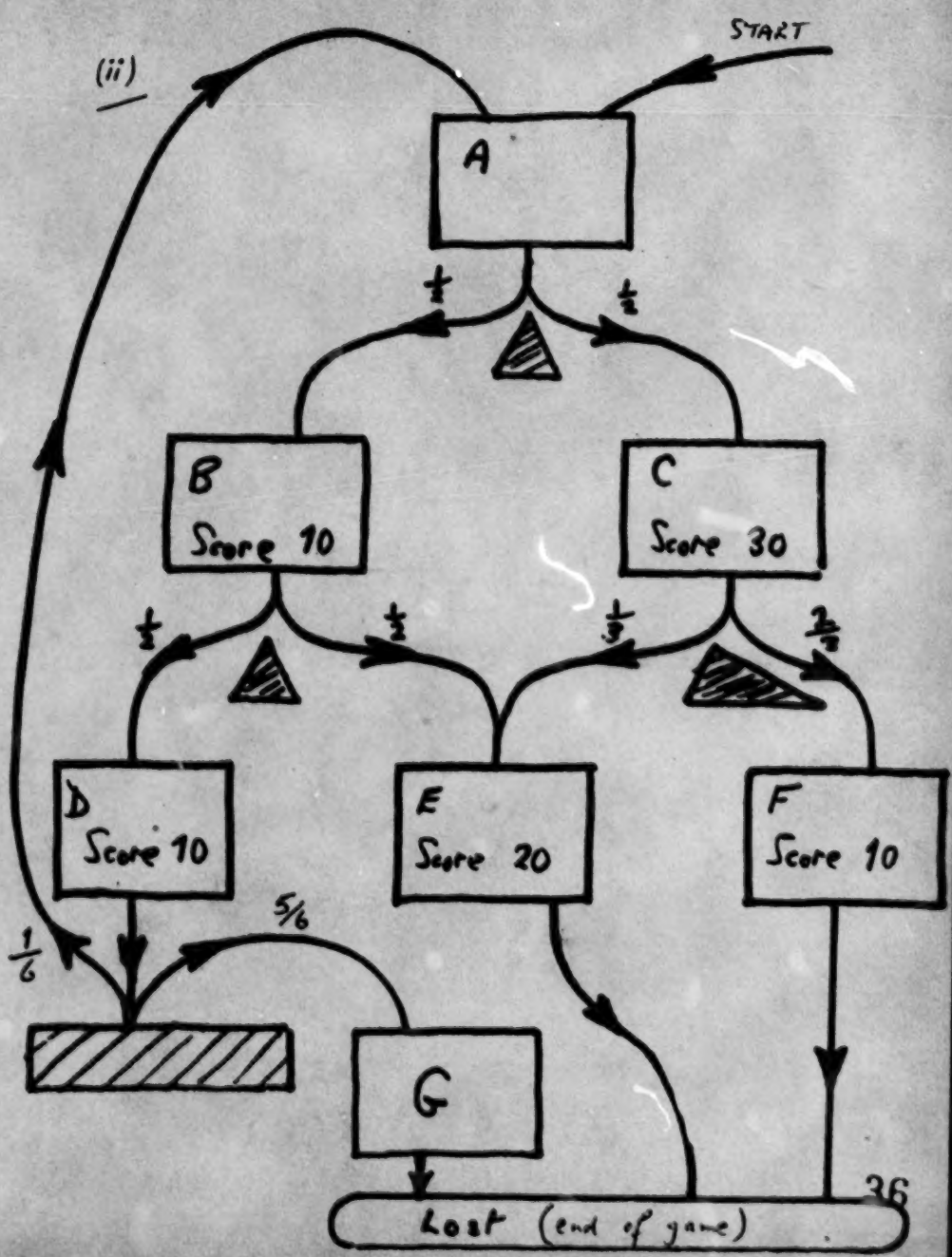


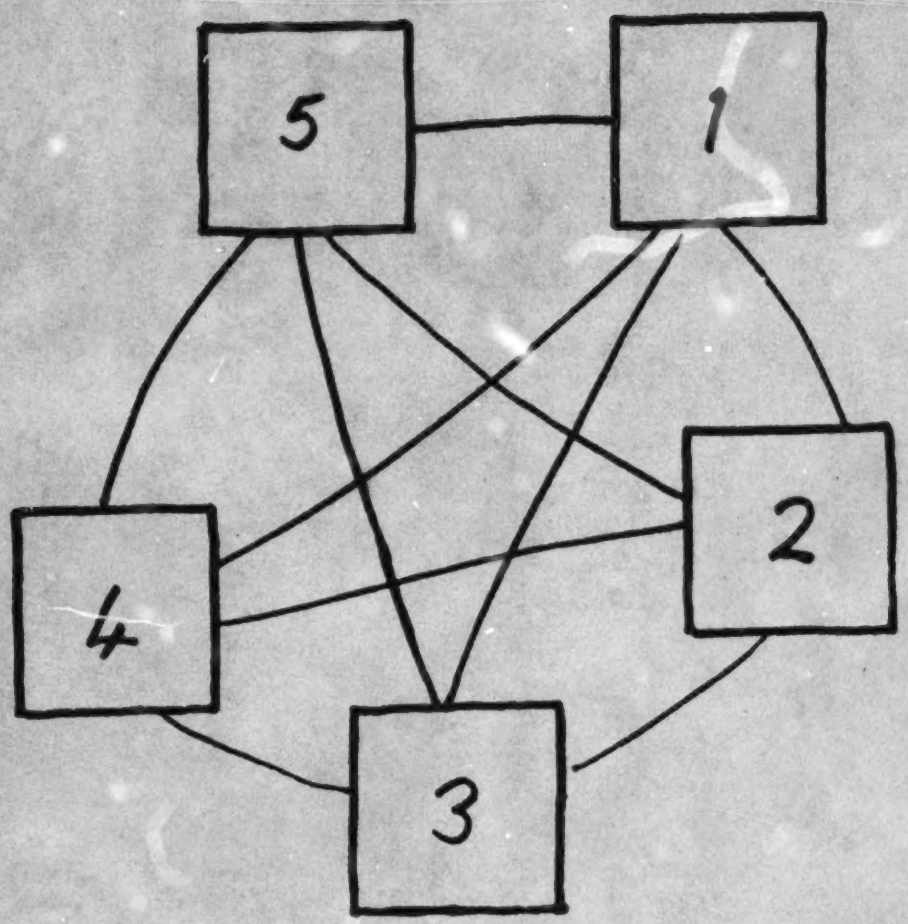
P_8



Worksheet 3(a)



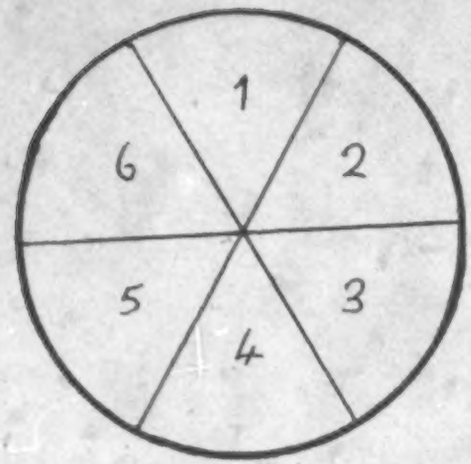




Worksheet 4(a)

1	2	3
4	5	6
7	8	9
10	11	12

Worksheet 5 (a)

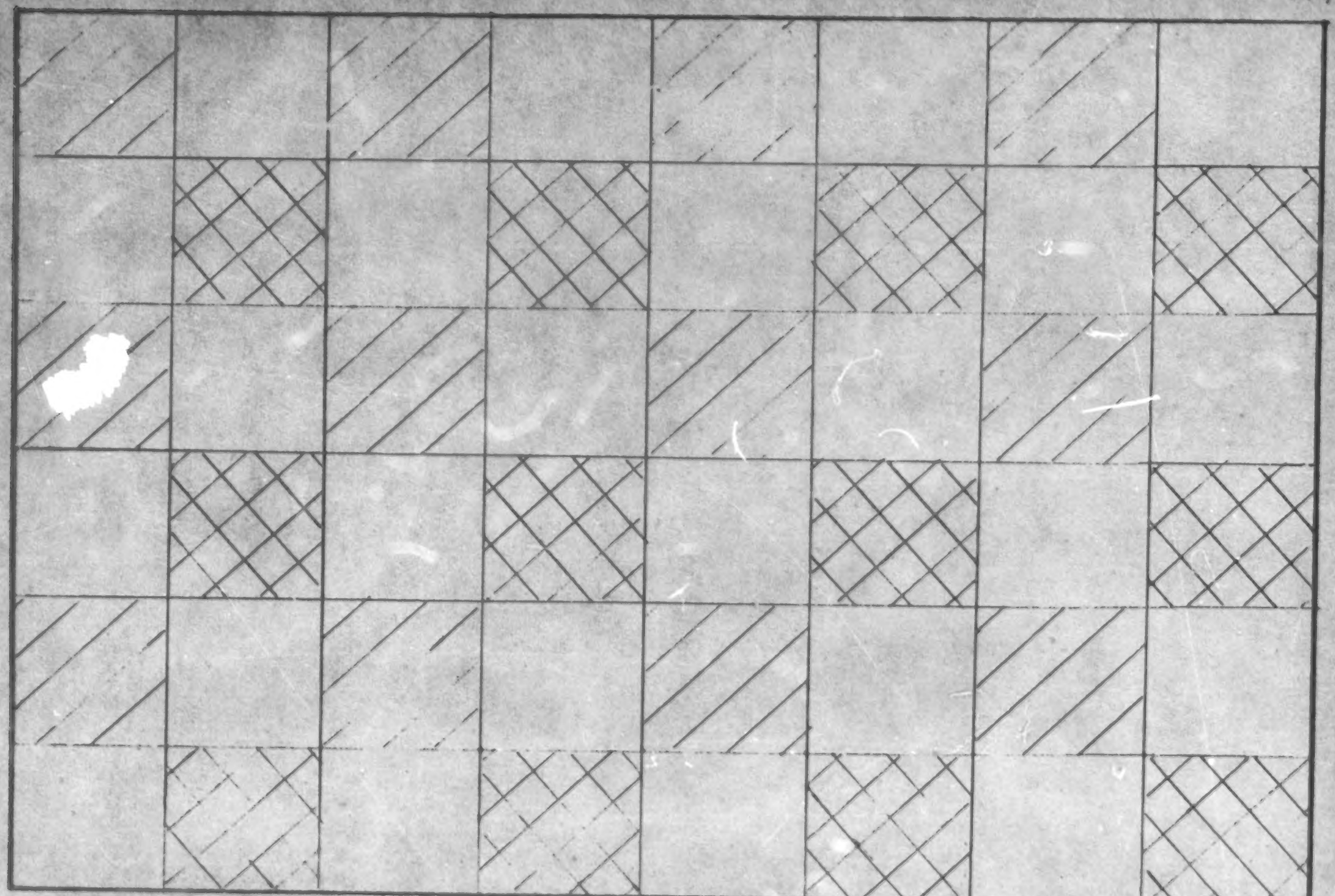


1	2	3	4	5
6	7	8	9	10

Worksheet 15 (a)

1	2	3	4	5	6	7	8	9
S				P				H

Worksheet 6 (a)/16a

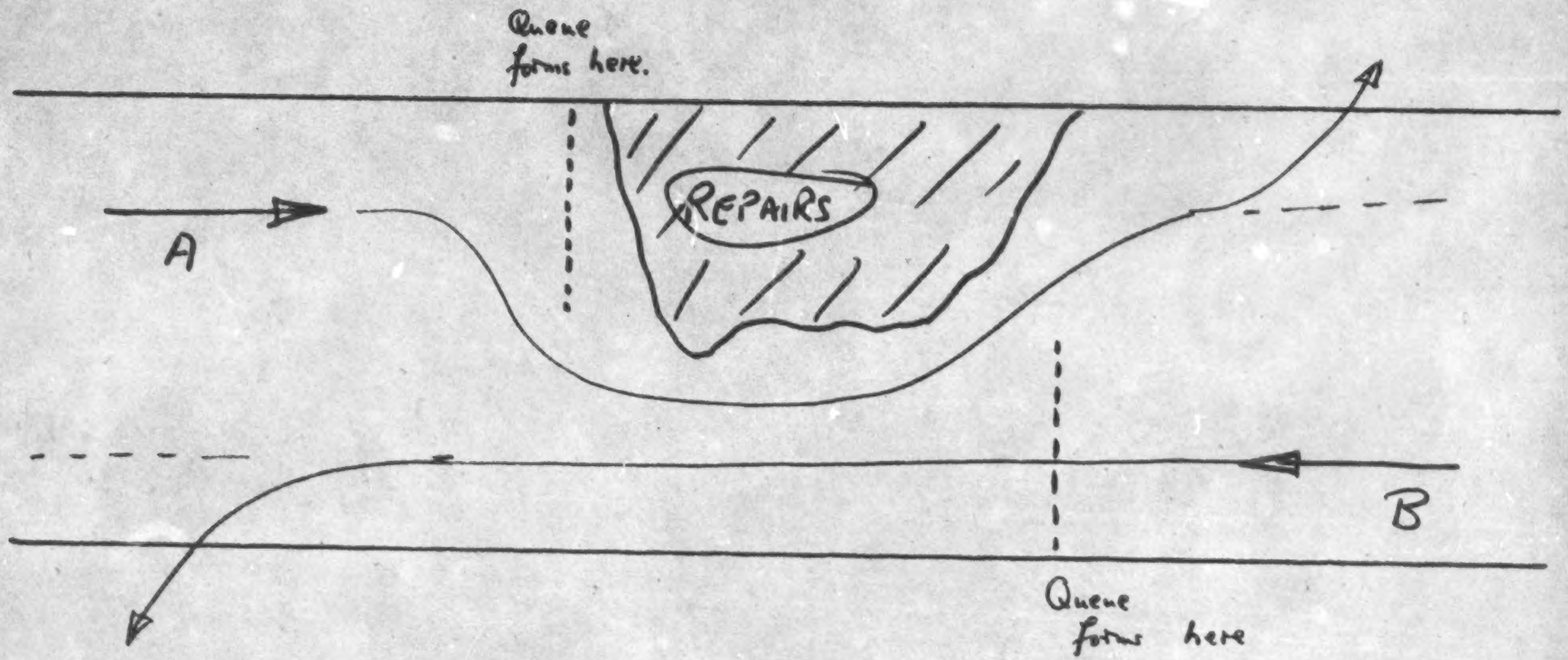


41

42

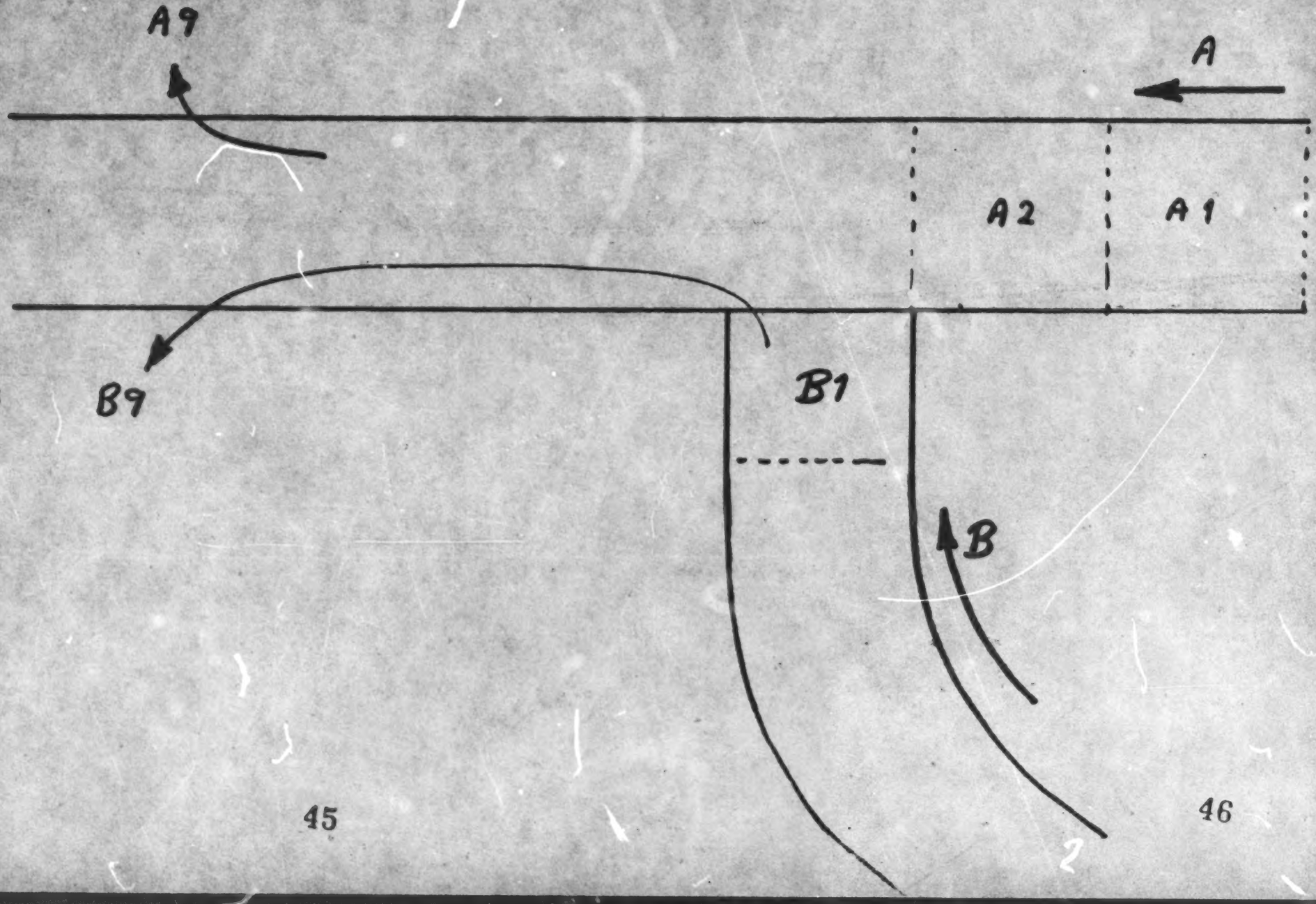
Worksheet 10(a)

Space for counters (vehicles) from A

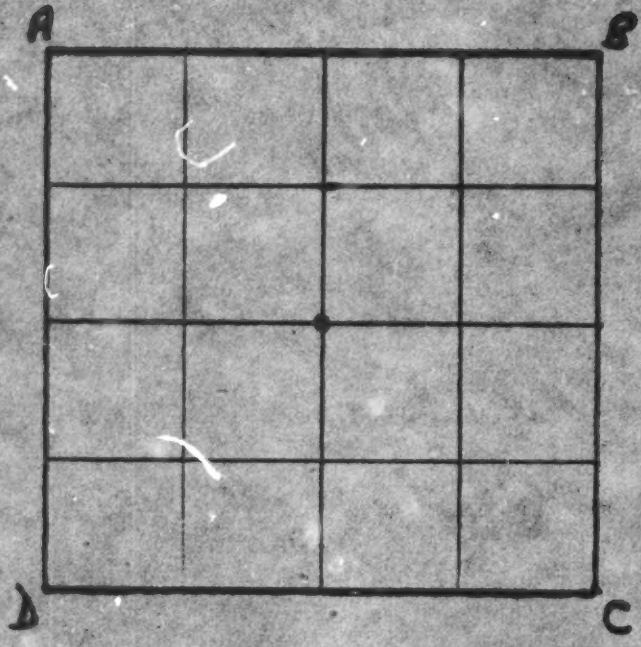


Space for counters (vehicles) from B

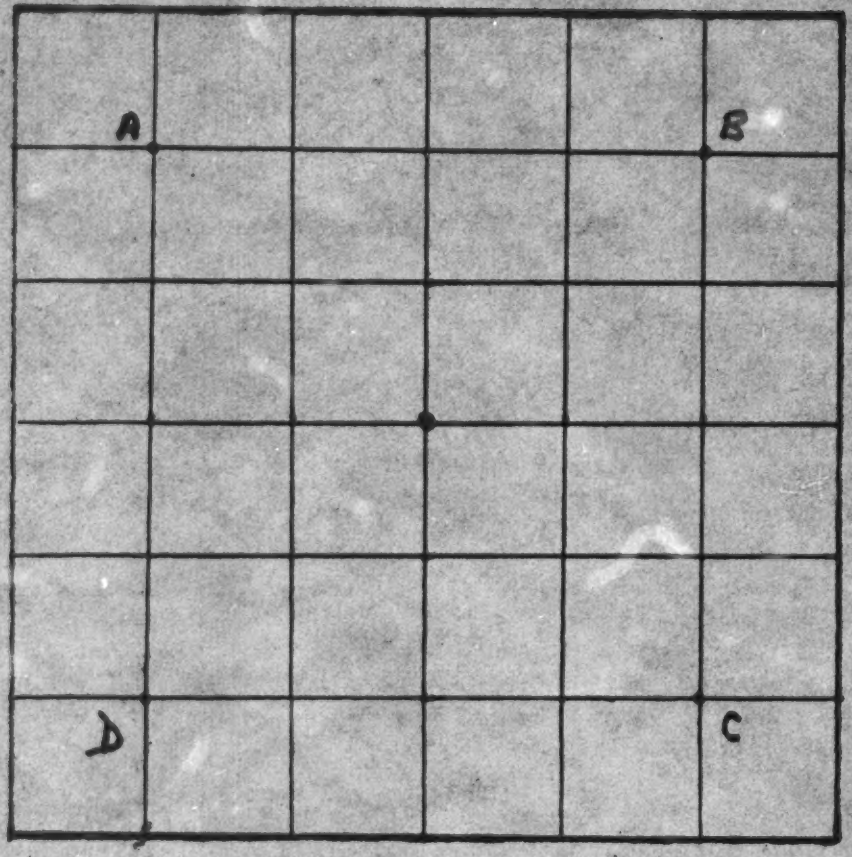
Worksheet 11(a)



Worksheet 13(a)



47



48

1st ↓ / 2nd →	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Worksheet 4(c)

Second die score

First die score	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

Third die score = 3

Second die score

First die score	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

Third die score = 4

Worksheet 4(b)

Second die score

First die score	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

Third die score = 1

Second die score

First die score	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

Third die score = 2

Second die score

First die score	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

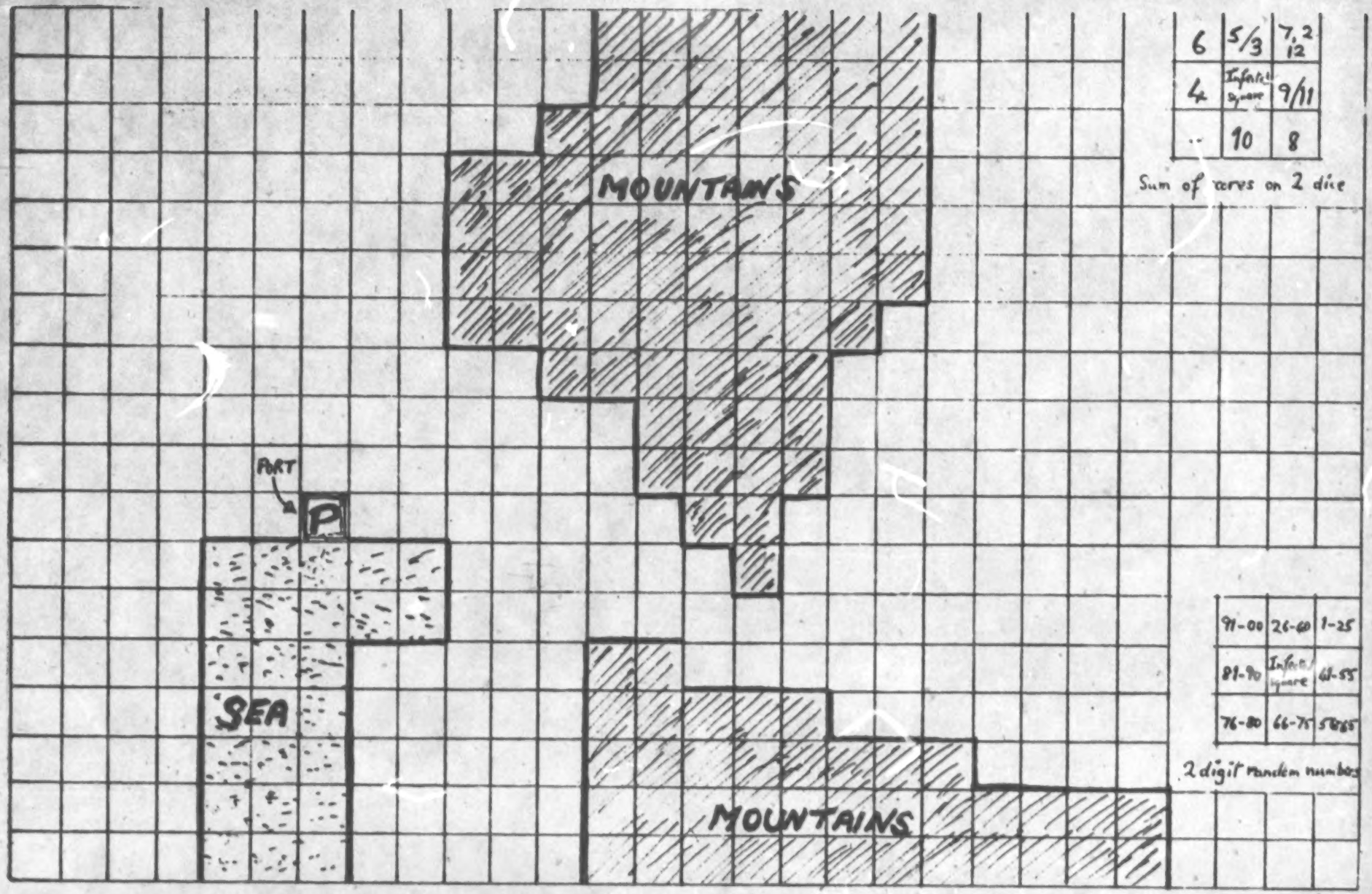
Third die score = 5

Second die score

First die score	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

Third die score = 6

Worksheet 17(a).



6	5/3	7, 2/12
4	Infantry ignore	9/11
..	10	8

Sum of scores on 2 dice

91-00	26-60	1-25
81-90	Infantry ignore	61-55
76-80	66-75	5685

2 digit random numbers

RANDOM DIGITS

49487	52802	28607	62058	78228	47041	18519	17889	45869	14454
48029	91539	46317	84803	86056	62812	33584	70391	77749	64906
25252	97738	23901	11106	86864	45808	22557	23214	15021	54268
02431	42103	96960	10620	29188	05863	92900	06836	13433	21709
69414	89353	70724	67893	23218	72452	03095	68333	13751	37260
77285	35179	92042	67581	67673	68374	71115	98166	43352	06414
52852	11444	71868	34534	69124	02760	06406	95234	37995	78560
93740	98054	30195	09891	18453	79454	01156	95522	06884	59073
85022	58736	12138	35146	62085	36170	25433	80787	96496	40579
17778	03840	16362	62595	08149	19001	67367	13138	02400	89515
83833	93449	57781	94621	90998	37561	59688	93299	27726	82167
63789	54958	33167	10309	40343	81023	61590	44474	39810	10305
61840	81740	09866	24981	15467	22494	38121	99025	86472	80921
42243	10153	20891	90883	15782	98167	68367	99166	92143	82441
45236	09129	53031	12260	01278	14404	40969	33419	14188	69557
40338	42477	78804	36272	72043	07958	57158	60979	79891	92409
54040	71253	88789	98203	54999	36564	00789	68879	46134	83941
49158	20908	44859	29089	61307	14425	44533	84909	37353	61137
80958	03808	83655	18415	96563	43582	82207	53322	30419	64435
07636	04876	61063	57571	69434	14965	20911	73162	33576	52839
37227	80750	08621	97048	60438	75053	05939	34414	16685	32103
99460	45915	45637	41353	35335	69087	57536	68418	10247	93253
60248	75845	72963	37833	42393	28185	31880	00241	31642	37526
95076	79089	87380	28982	97750	82221	35584	27484	85793	69755
09442	97852	26586	32796	51513	47475	48621	20067	88975	39506
30458	49207	62358	45532	30047	53016	10375	97204	98675	77634
38905	91282	79309	49022	17405	18830	09166	07629	01785	78317
96545	15638	90114	93730	13741	70177	49175	42113	21600	69625
21944	28328	00692	89164	96025	01383	50252	67044	70596	48266
36910	71928	63326	00980	32154	46006	62289	28079	03076	15619
38745	47627	28856	28382	60639	51370	70091	58261	70135	88259
32519	91993	59364	83994	59873	51217	62806	20028	26545	16820
75757	12965	29285	11481	31744	41754	2442	80809	02354	37895
06911	97756	89551	27464	25133	50026	16436	75846	83618	08533
89867	03328	76911	93168	56236	39056	67905	94933	05456	52347
30432	99488	75363	94187	32385	23837	10872	22793	26232	87356
68552	55201	33946	42495	28384	89889	50278	91985	58185	19124
22403	56698	88524	13692	55012	25343	76391	48029	72278	58586
70701	36907	51242	52083	43126	90379	60380	98513	85596	16528
69804	96122	42342	28467	79037	13218	63510	09071	52438	24850
65806	22398	19470	63643	27055	02706	33464	65384	02613	81668
43902	53060	54319	19347	59506	75440	90826	53642	23829	76236
14547	48771	73152	62440	15770	03281	58124	09533	43722	03856
47363	36295	62126	42358	20322	82000	52830	93540	13284	96496
26244	87033	90246	79131	38773	67687	45541	54975	17508	18367
72875	39496	06385	48458	20545	74383	22814	36752	10606	48774
09065	16283	61398	08288	00708	21816	39615	03102	02834	04116
68256	51225	92645	77747	33104	81206	00112	53445	04212	58476
38744	81018	41909	70458	72459	66136	97266	26490	08771	45022
44375	15 19	35750	59924	82429	30288	61064	26489	87001	84273